Here is the sketch of a proof that there is a solution to the August, 2018, IBM Ponder This Challenge for which all 18 integers are prime.

Suppose that $W = \langle W_1, W_2, W_3, ..., W_9 \rangle$ and $G = \langle G_1, G_2, G_3, ..., G_9 \rangle$ are vectors with real components, that $A = \frac{W_1 + W_2 + W_3 + \cdots W_9}{10}$, and that the following three conditions are satisfied:

(i) The component sums of the two vectors are equal , i.e. $\sum_{i=1}^{9} W_i = \sum_{i=1}^{9} G_i$;

(ii) No two elements of the set $\{W_1, W_2, ..., W_9, G_1, G_2, ..., G_9\}$ are equal;

(iii) The vector $\langle W_1 - A, W_2 - A, ..., W_9 - A \rangle$ is a scalar multiple of G , i.e. $\langle W_1 - A, W_2 - A, ..., W_9 - A \rangle = \lambda \langle G_1, G_2, ..., G_9 \rangle$ for some real λ .

Let f be the function $f(x) = px + q$ for real p , q , and x . Then the vectors
$W_f = \langle f(W_1), f(W_2), ..., f(W_9) \rangle = \langle pW_1 + q, pW_2 + q, ..., pW_9 + q \rangle$ and
$G_f = \langle f(G_1), f(G_2), ..., f(G_9) \rangle = \langle pG_1 + q, pG_2 + q, ..., pG_9 + q \rangle$ and the number
$A_f = \frac{(pW_1 + q) + (pW_2 + q) + \cdots + (pW_9 + q)}{10}$ satisfy conditions (i), (ii) and (iii) above, namely:

(i) The component sums of W_f and G_f are equal;

(ii) No two of the eighteen components of W_f and G_f are equal;

(iii) The vector $\langle f(W_1) - A_f, f(W_2) - A_f, ..., f(W_9) - A_f \rangle$ is a scalar multiple of G_f .

The first two conditions are immediately seen to be true; the third requires a little bit of algebra, which in the fine tradition of mathematical exposition we omit.

The vector W encodes the numbers of drops of water the humans start with, and the vector $\langle W_1 - A, W_2 - A, ..., W_9 - A \rangle$ encodes how many drops they give up so that all ten (the nine humans plus Sauron) have the same number A of drops of water at the end. Our result states that *if we disregard the conditions on integers and primality*, then any solution to the Challenge produces other solutions by linear transformation.

Now let $W^* = \langle 45, 46, 47, 48, 49, 50, 51, 52, 58 \rangle$ and $G^* = \langle 4, 14, 24, 34, 44, 54, 64, 74, 134 \rangle$. It is easy to verify that these vectors satisfy conditions (i) and (ii) and (iii) and thus provide a solution to the Challenge if we ignore the primality condition. Next we use the argument above and find a linear transformation that will produce a solution with eighteen primes.

It is known that there exists a sequence $p_1, p_2, p_3, ..., p_{131}$ of 131 primes in arithmetic progression. (See Sloane, The On-Line Encyclopedia of Integer Sequences A005115 for the reference to the article by Ben Green and Terence Tao which proves this fact. Note that we are not claiming that the primes are consecutive.) The set $\{4,5,6,...,134\}$ contains all the elements of the vectors W^* and G^*. There is a linear transformation f which maps the 131 integers in this set $\{4,5,6,...,134\}$ onto the set $\{p_1, p_2, p_3,..., p_{131}\}$, so that $f(4) = p_1$, $f(5) = p_2$, ..., $f(134) = p_{131}$. (This is true since the elements of both $\{4,5,6,...,134\}$ and $\{p_1, p_2, p_3,..., p_{131}\}$ are in arithmetic progression.) By our first result, the vector $f(W^*)$ will be a complete solution to the Challenge.

It is noted in the discussion in Sloane's A005115 that Green and Tao proved that the last prime p_n in a sequence of n primes p_1, p_2, ..., p_n in arithmetic progression satisfies the inequality
$p_n < 2\text{^}2\text{^}2\text{^}2\text{^}2\text{^}2\text{^}2\text{^}2\text{^}O(n)$.

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