TWO TRIANGLES

Let $\triangle ABC$ be the outer triangle, and $\triangle DEF$ be the inner one, so that $D \in BC$, $E \in AC$, $F \in AB$. Denote $\alpha = \angle FDB$, $\beta = \angle EFA$, $\gamma = \angle DEC$, and assume without loss of generality that

$$\alpha \geq \beta \geq \gamma. \tag{0.1}$$

Denote $\alpha' = \angle DFB$, $\beta' = \angle FEA$, $\gamma' = \angle EDC$. Clearly, $\alpha' = 2\pi/3 - \beta$, $\beta' = 2\pi/3 - \gamma$, $\gamma' = 2\pi/3 - \alpha$, and hence

$$\beta' \geq \alpha' \geq \gamma'. \tag{0.2}$$

Consider three triangles: $\triangle BFD$, $\triangle AEF$, $\triangle CDE$. Let us superimpose the bases FD, EF, and DE with a horizontal segment PQ. Then the vertices A, B, C lie on a upper semicircle centered at Q. Let R be the tangent point of the line through P and the semicircle; R divides the semicircle into two arcs: the front one (which can be seen from P) and the back one.

There are two cases:

1) all angles of $\triangle ABC$ are acute;

2) one of the angles of $\triangle ABC$ is obtuse.

In the first case, all three points A, B, C lie on the back arc. By (0.1), their clockwise order along the semicircle is $RCAB$, which contradicts (0.2) unless $\alpha = \beta = \gamma$.

In the second case, one point lies on the front arc. By (0.1), this point is C. Denote by C' the second intersection point of the line through P and C with the semicircle; by (0.2), the clockwise order of the points along the semicircle is $CRABC'$.

Clearly, $\angle PCQ + \angle PC'Q = \pi$. On the other hand, points A and B lie inside the circle through P, Q, and C', and hence $\angle PAQ \geq \angle PC'Q$, $\angle PBQ \geq \angle PC'Q$. Consequently, $\angle PCQ + \angle PAQ + \angle PBQ > \pi$, a contradiction.