A Simple Solution to IBM Ponder This August 1998 Challenge

Nan Zhong
zhongn@ethz.ch

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There are three cases of sizes of $\angle A$, $\angle B$ and $\angle C$: (1) All are equal; (2) Two are equal but unequal to the third one; (3) Pairwise unequal. Our solution proves that in the first case the $\triangle ABC$ is equilateral and derives the other two cases to contradictions.

The first case is trivial. $\angle A = \angle B = \angle C$, $\triangle ABC$ is equilateral.

The second case is impossible. WLOG, assume $\angle A \neq \angle B = \angle C$. Since $\angle B = \angle C$, $AC = AB$. Given $CF = AD$, $AF = BD$. $\triangle DEF$ is equilateral, $DF = ED$. Therefore $\triangle ADF$ is congruent to $\triangle BED$. As a result, $\angle A = \angle B$, contradicting our assumption.

The third case is impossible. WLOG, assume $\angle B > \angle A$ and $\angle B > \angle C$. Draw two auxiliary line segments perpendicular to the two sides of $\angle B$ separately: EG over BA at G; and FH over BC at H.

Given $\angle A + \angle B + \angle C = 180°$, $\angle B > 60°$. $\angle B > \angle C$ and $\angle B < 180° - \angle C$, so that $\sin B > \sin C$. Since $\angle B > \angle C$ and $BE = CF$, $EG = BE \cdot \sin B > CF \cdot \sin C = FH$. $\triangle DEF$ is equilateral, $ED = FE$, which leads to $\sin \angle EDG = EG/ED > FH/FE = \sin \angle FEH$. Either $\angle EDG > \angle FEH$ or $\angle EDG > 180° - \angle FEH$. Considering $\angle EDG$ and $\angle FEH$ are both in straight triangles, $\angle EDG > \angle FEH$. $90° = \angle EDG + \angle DEG > \angle FEH + \angle DEG$. $\angle BEG = 180° - \angle FEH - \angle DEG - \angle DEF > 180° - 90° - 60° = 30°$. However, $\angle DEF = 90° - \angle B < 90° - 60° = 30°$, contradicting our assumption.

This completes the proof.