**Part 1 & 2**. For any ABC - primitive right triangle with integer sides which form a primitive Pythagorean Triple we have:

$(2pq, p^2 - q^2, p^2 + q^2)$ , where GCD( $p,q$ ) = 1, $p > q$ .

Legs are "one odd other even" while hypotenuse is always odd in ABC, because $p$ and $q$ are "one odd other even".

DEF - isosceles triangle, let DE=DF and EF-basis, G-middle of EF such that EG=GF. DEG – right triangle in G, $DE^2 = EG^2 + GD^2$ , DE-integer. Also EG-integer because EF-even integer. Indeed if EF-odd integer, $EF^2 = 4k+1$ , $4DE^2 = EF^2 + 4GD^2$, then square of 2GD, $(2GD)^2 = 4GD^2 = 4l-1$, contradiction. $S_{DEG} = \frac{S_{DEF}}{2} = \frac{EG \cdot GD}{2}$ and $S_{DEF}$ - integer then GD-rational and $GD^2$-integer as $GD^2 = DE^2 - EG^2$, so GD-integer.

We conclude that DEG-right triangle with integer sides (not necessarily relatively prime), which form also a Pythagorean Triple:

$(2rsd, (r^2 - s^2)d, (r^2 + s^2)d)$ , where GCD( $r,s$ ) = 1, $r > s$, $r$ and $s$ - "one odd other even" and $d$ is GCD of integers which represent the sides of triangle DEG.

Condition **(1)** to have the same perimeter for ABC and DEF:

$P_{ABC}$ = AB+BC+CA = $2p(p+q)$

$P_{DEF} = 2$ (DE+EG) , DE = $(r^2 + s^2)d$

Case **A**. Leg EG of the form $(r^2 - s^2)d$ , $P_{DEF} = 4r^2 d$ then:

$p(p+q) = 2r^2 d$

Case **B**. Leg EG of the form $2rsd$ , $P_{DEF} = 2(r+s)^2 d$ then:

$p(p+q) = (r+s)^2 d$

Condition **(2)** to have the same area for the triangles:

$S_{ABC} = pq(p^2 - q^2)$

$S_{DEF} = 2S_{DEG} = 2rs(r^2 - s^2)d^2$

$pq(p^2 - q^2) = 2rs(r^2 - s^2)d^2$

Our initial problem became equivalent with solving a system of equations in each of the two cases, by considering initial conditions imposed for $p, q, r, s$:

**Case A**. (1) $p(p+q) = 2r^2 d$

(2) $p(p+q)q(p-q) = 2rs(r^2 - s^2)d^2$

Dividing (2) by (1) , we get $q(p-q) = \frac{s(r^2 - s^2)}{r} d$ - integer. GCD( $r, s(r^2 - s^2)$ ) = 1 so $r \mid d$, $d = rt$ . GCD( $p(p+q), q(p-q)$ ) = 1 then $t$ =1, $d = r$ and system **(A)** is:

**(1')** $p(p+q) = 2r^3$

**(2')** $q(p-q) = s(r^2 - s^2)$ , then:

$2 \mid p$, $2 \mid r$ and $16 \mid p$, so taking into account prime relativity:

$p = 2u^3$ even , $p+q = v^3$ odd ,

$q = v^3 - 2u^3$ odd , $p-q = 4u^3 - v^3$ odd ,

$r = uv$ even , $s$ odd ,

$u$ even , $v$ odd , GCD($u,v$)=1 ,

$s$ has to satisfy:

$$(v^3 - 2u^3)\,(4u^3 - v^3) = s\,(u^2v^2 - s^2)\ ,\ \text{GCD}(s,uv)=1\ ,\ 0 < s < uv \quad \textbf{(*)}$$

$u < v$ and "$u$ even, $v$ odd" " $\Rightarrow$ $u \ge 2, v \ge 3$, $2u^3 < v^3 < 4u^3$.

$$-8u^6 + 6u^3v^3 - v^6 = s\,(u^2v^2 - s^2) \quad \Leftrightarrow \quad -8u^6 + 6u^3v^3 - v^6 = s\,(uv\text{-}s)\,(uv\text{+}s) \ \Rightarrow$$

$$8u^6 + v^6 \equiv s^3 \pmod{u^2v^2} \quad \Leftrightarrow \quad (2u^2 + v^2)^3 \equiv s^3 \pmod{u^2v^2}\ , \ \text{then:}$$

$$s \equiv a(2u^2+v^2) \pmod{uv}\ ,\quad a^3 \equiv 1 \pmod{uv}\ ,\ 0 < a < uv\ ,\ \text{so:}$$

$$s = a(2u^2+v^2) - buv\ ,$$

where GCD($a$,$uv$)= 1 else $a^3 \ne 1 \pmod{uv}$, GCD($uv$, $2u^2+v^2$)= 1 .

Consider $f(s) = s\,(u^2v^2 - s^2) - (-8u^6 + 6u^3v^3 - v^6)$ .

For "given" fixed integers $u$, $v$, if exists such values $s$ odd, $0 < s < uv$, that satisfy **(*)**, there are at most 2 of them (from Viete relationship $s_1 + s_2 + s_3 = 0$, at least one root is negative). If exists exactly 2 values then 3$^{rd}$ is a negative even integer $s_3 = -s_1 - s_2$ and $f(s_3)$ will be odd, contradiction with $s_3$ - root. There is at most one such value $s$.

The "linear Diophantine equation" in 2 variables X, Y integers:

$$c\text{X} - d\text{Y} = s\ ,$$

where GCD($c$,$d$)= 1 and GCD($c$,$s$)= 1, $c$,$d$,$s$ integers, always has exactly one solution $(X_0, Y_0)$ for $0 < \text{X} < d$.

Consider such a "linear Diophantine equation" in 2 variables $a$, $b$, where $u$, $v$, $s$ are considered as "given" fixed integers, GCD($s$,$uv$)=1, $0 < s < uv$:

$$(2u^2+v^2)\,a - uv\,b = s\ ,$$

where GCD($uv$, $2u^2+v^2$)= 1 and GCD($s$,$uv$)= 1. This equation has an unique solution $(a_0, b_0)$ that satisfy $0 < a_0 < uv$.

We have to find such possible values for "variables" $(a_0, b_0)$ that satisfy in addition $a_0^3 \equiv 1$ (mod $uv$). $a_0 = 1$ and $b_0 = 2$ is such a solution:

$$2uv < 2u^2 + v^2 \quad \Leftrightarrow \quad 0 < \frac{1}{2}(2u - v)^2 + \frac{1}{2}v^2\ , \text{true}\ ,$$

$$2u^2 + v^2 < 3uv \quad \Leftrightarrow \quad (u - v)\,(2u - v) < 0\ , \text{true}\ (2u^3 < v^3 < 4u^3 \ \Rightarrow\ u < v < 2u)\ .$$

This is the only solution for "variables" $a$, $b$. Indeed the subset of the set of residues for this particular $uv$-even, $Z_{uv}$, denoted $Z_{uv}^*$, that contains only numbers relatively prime with $uv$, is a multiplicative abelian group of order $\phi(uv)$ - obviously the subset and multiply exhibit closure, while inversability occurs from GCD($s$,$uv$)=1, for $0 < s < uv$, and the existence of an unique inverse is guaranteed by this type of "linear Diophantine equation" . Then:

$$s = (2u^2+v^2) - 2uv\ .$$

Substitute in equation **(*)** we get homogeneous polynomials in 2 variables: $u$ , $v$ – of degree 6 with integer coefficients. Simplify $-8u^6$, $-v^6$ and give $uv$ as common factor, then results another homogeneous polynomial of degree 4 equal to 0, having the term $6v^4$, while all the others contain $u$:

$$uv\,[u\,(24u^3 - 34u^2v + 24uv^2 - 17v^3) + 6v^4] = 0\ ,$$

so $u \mid 6v^4 \ \Rightarrow\ u \mid 6$, $u$ even $\Rightarrow$ $\boldsymbol{u = 2}$ **,** $\boldsymbol{v = 3}$ . The only solution in this case is:

$p = 16$ , $q = 11$ , $r = 6$ , $s = 5$ (so $d = 6$):

**lengths of the sides of ABC – primitive right triangle: ( 352, 135, 377)** ,

lengths of the sides of DEG - right triangle: ( 360, 66, 366) ,

**lengths of the sides of DEF – isosceles triangle: DE=DF=366 , EF=132** ,

indeed $P_{ABC} = P_{DEF} = \mathbf{864}$ , $S_{ABC} = S_{DEF} = \mathbf{23760}$ .

The system **(A)** has this unique solution.

**Case B**. (1) $p(p+q) = (r+s)^2 d$

(2) $p(p+q)q(p-q) = 2rs(r^2 - s^2)d^2$

Analog, $q(p-q) = \dfrac{2rs(r-s)}{r+s} d$ - integer. GCD( $r, s(r^2 - s^2)$ ) = 1 so ($r$+$s$)| $d$ , $d$=($r$+$s$)$t$ .

GCD( $p(p+q), q(p-q)$ ) = 1 then $t$ =1, $d$ = $r$+$s$ and system **(B)** is:

**(1')** $p(p+q) = (r+s)^3$

**(2')** $q(p-q) = 2rs(r-s)$

again taking into account prime relativity:

$p = u^3$ odd , $p+q = v^3$ odd ,

$q = v^3 - u^3$ even , $p-q = 2u^3 - v^3$ odd ,

$r + s = uv$ odd , $r$ and $s$ - "one odd other even"

$u$ odd , $v$ odd , GCD($u$,$v$)=1 ,

$s$ has to satisfy:

$$(v^3 - u^3)(2u^3 - v^3) = 2s(uv-s)(uv-2s) \text{ , GCD}(s,uv)=1 \text{ , } 0 < s < \frac{uv}{2} \quad \textbf{(**)}$$

$u < v$ and "$u$ odd, $v$ odd" " $\Rightarrow$ $u \geq 3, v \geq 5$, $u^3 < v^3 < 2u^3$.

$-2u^6 + 3u^3v^3 - v^6 = 2s(uv-s)(uv-2s) \Rightarrow$

$2u^6 + v^6 \equiv -4s^3 \pmod{uv}$

Exists an unique determined residue $m$ in $Z^*_{uv}$ ($uv$-odd), such that $m^3 \equiv 2 \pmod{uv}$ while $n$ is it inverse , $mn \equiv 1 \pmod{uv}$, $n^3 \equiv \dfrac{uv+1}{2} \pmod{uv}$, GCD($m$,$uv$)= 1, GCD($n$,$uv$)= 1, $m, n > 1$. So:

$(-m^2 s)^3 \equiv m^3 u^6 + v^6 \pmod{uv} \Rightarrow (-m^2 s)^3 \equiv (mu^2 + v^2)^3 \pmod{uv} \Rightarrow$

$s^3 \equiv -(n^2(mu^2 + v^2))^3 \pmod{uv} \Rightarrow$

$s \equiv -a(n^2(mu^2 + v^2)) \pmod{uv}$ , $a^3 \equiv 1 \pmod{uv}$, $0 < a < uv$ , then:

$s = -an^2(mu^2 + v^2) + buv = -an(u^2 + nv^2) + cuv$ .

Consider $g(s) = 2s(uv-s)(uv-2s) - (-2u^6 + 3u^3v^3 - v^6)$ ,

that has at most one integer root $0 < s < \dfrac{uv}{2}$ , $s_1 + s_2 + s_3 = 3\dfrac{uv}{2}$. If 2 such odd integers exists, for the 3$^{rd}$ root, $g(s_3)$ will be again odd, contradiction.

Regarding our all conditions of prime relativity (in multiplicative group $Z^*_{uv}$), we proof in the same manner as in case **A,** by developing another "linear Diophantine equation" in 2 variables $a$, $b$. For $0 < a < uv \Rightarrow a_0 = 1$ . Then:

$s = -n(u^2 + nv^2) + c_0 uv$ ,

where $c_0 = \left[ \dfrac{n(u^2 + nv^2)}{uv} \right]$.

Replace in equation **(**)**. Results another homogeneous polynomial in 2 variables: $u$ , $v$ - of degree 6 equal to 0, having the following coefficients for the terms:

$u^6$ : $4n^3 - 2$

$u^5 v$ : $-6n^2(2c_0 - 1)$

$u^4 v^2$ : $12n^4 + 2n(6c_0^2 - 6c_0 + 1)$

$u^3 v^3$ : $-12n^3(2c_0 - 1) - 2c_0\ (c_0 - 1)(2c_0 - 1) + 3$

$u^2 v^4$ : $12n^5 + 2n^2(6c_0^2 - 6c_0 + 1) = n\ (12n^4 + 2n(6c_0^2 - 6c_0 + 1))$

$uv^5$ : $-6n^4\ (2c_0 - 1) = n^2(-6n^2(2c_0 - 1))$

$v^6$ : $4n^6 - 1 = n^3(4n^3 - 2) + 2n^3 - 1$

Rewriting:

$$(4n^3 - 2)\,(u^2 + nv^2)^3 - 6n^2(2c_0 - 1)\,(u^2 + nv^2)^2\,uv + 4n(3c_0^2 - 3c_0 + 2)\,(u^2 + nv^2)\,u^2v^2 -$$

$$- (2c_0 - 3)(2c^2 + 1)\ u^3v^3 + (2n^3 - 1)\ v^6 = 0\,,$$

and takes into account GCD(($u^2$+$nv^2$),$uv$)=1 one can get (mod($u^2$+$nv^2$)):

$$- (2c_0 - 3)(2c_0^2 + 1)\ u^3 + (2n^3 - 1)\ v^3 \equiv 0\,, \text{ then:}$$

$$n(2c_0 - 3)(2c_0^2 + 1) + (2n^3 - 1) \equiv 0\,,$$

contradiction with $c_0 = \left[ \dfrac{n(u^2 + nv^2)}{uv} \right]$.

We conclude there is no such $s$ , GCD($s$,$uv$)=1 , $0 < s < \dfrac{uv}{2}$ , that satisfy equation **(**)**.

The system **(B)** has no solution.

We conclude that both systems have no solution for $\boldsymbol{u} \geq \mathbf{3}$.

The proof for the uniquely of the solution found is done.